Apr 2: Cubic equations continued

Outline

- History
- Recall cubic solution
- Examples
- Structure of roots & symmetric functions

§1. History of cubics

Characters

- Luca Pacioli
  1494 "Summa de aritmetica..."
  Cubic cannot be solved!
- Scipione $\boxed{\text{del Ferro}}$
  - Solved the cubic in 1515
  - Kept the solution private
  - share with his student Fior prior to his death in 1526
- Antonio Fior
  - good but not great
  - couldn't keep the secret
- Niccolo Tartaglia
  - independently found the soln
- Girolamo Cardano (Cardan)
  - invite Fior to visit
  - 1st latin treatise on algebra 1545 (Ars Magna)

Considered two cases

$\boxed{x^3+ax+b=0}$ $\qquad x^3=ax+b$

Didn't use negative numbers!

del Ferro

Baby case

Solve $x^3=a$ a real number

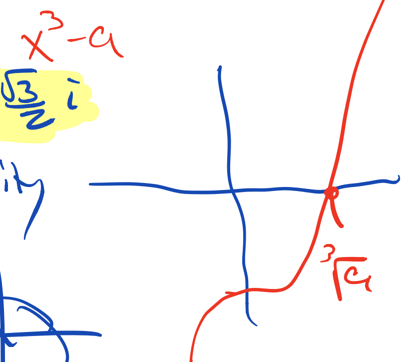
$\sqrt[3]{a}$ is a solution

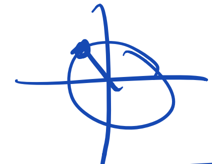
unique real number whose cube is a

Let $\omega = e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$

primitive 3rd root of unity

$\omega^3 = 1$





x^3-a has 3 solutions

$\sqrt[3]{a}, \omega\sqrt[3]{a}, \omega^2\sqrt[3]{a}$

§2. Cubic soln

- First reductions led us to

$$x^3 + a_1 x + a_0 = 0$$

- New trick: $\underbrace{x = y - \frac{a_1}{3y}}_{\text{non-linear substitution}}$

Idea: First solve for y & then solve for x.

- Substituting in x leads to

$$y^6 + a_0 y^3 - \frac{a_1^3}{27} = 0$$

Solve for y^3

$$y^3 = -\frac{a_0}{2} \pm \sqrt{\frac{a_0^2}{4} + \frac{a_1^3}{27}}$$

$$\boxed{y = \omega^i \sqrt[3]{-\frac{a_0}{2} \pm \sqrt{\frac{a_0^2}{4} + \frac{a_1^3}{27}}}}$$

$i = 0, 1, 2$

Rmk: Gives 6 solns.

But actually only 3!

Rmk: If we substitute $x = y - \frac{a_1}{3y}$, get "Cardano's equation"

$$x = \sqrt[3]{-\frac{a_0}{2} + \sqrt{\frac{a_0^2}{4} + \frac{a_1^3}{27}}} + \sqrt[3]{-\frac{a_0}{2} - \sqrt{\frac{a_0^2}{4} + \frac{a_1^3}{27}}}$$

⚠ Don't apply naively!

Each cube root has 3 choices

$9 = 3 \cdot 3$ possible expressions

Rmk: Check solutions!

§3 Examples

Ex 1 $\quad x^3 + a_0 = 0 \quad (a_1 = 0)$

$\omega = e^{2\pi i/3}$

Know solns are

$$x = \sqrt[3]{-a_0},\ \omega\sqrt[3]{-a_0},\ \omega^2\sqrt[3]{-a_0}$$

Take $\quad x = y - \frac{a_1}{3y} = y$

$$y^3 = \frac{-a_0}{2} \pm \sqrt{\frac{a_0^2}{4} + \frac{a_1^3}{27}}$$

$$= \frac{-a_0}{2} \pm \sqrt{\frac{a_0^2}{4}}$$

$$\boxed{y^3 = -a_0} \quad \text{or} \quad 0$$

But $y=0$ is a not a soln!

$$\rightarrow \quad x = y = \sqrt[3]{-a_0},\ \omega\sqrt[3]{-a_0},\ \omega^2\sqrt[3]{-a_0}$$

Ex 2 $x^3 - 3x = 0$ $\quad a_1 = -3$, $a_0 = 0$

Again, know solns!

$x = y + \frac{1}{y}$

$y^3 = \pm\sqrt{-1} = \boxed{\pm i}$ $\quad i$, $-i$

3 cases for y if $y^3 = i = e^{\pi i/2}$

(1) $y = e^{\pi i/6} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

$\rightsquigarrow \boxed{x = e^{\pi i/6} + e^{-\pi i/6}}$

$= \left(\frac{\sqrt{3}}{2} + \cancel{\frac{1}{2}i}\right) + \left(\frac{\sqrt{3}}{2} - \cancel{\frac{1}{2}i}\right) = \sqrt{3}$

(2) $y = \omega e^{\pi i/6}$

$x = -\sqrt{3}$

(3) $y = \omega^2 e^{\pi i/6}$

$x = 0$

3 cases for y if $y^3 = -i = e^{3\pi i/2}$

(1) 0

(2) $-\sqrt{3}$

(3) $\sqrt{3}$

List 3 solns!

Beware: "Cardano's formula"

gives

$\boxed{\sqrt[3]{i} + \sqrt[3]{1}}$ not a soln